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SPACE WAVES ON THE SOURCE OF A FILM FLOWING DOWN THE SURFACE OF A VERTICAL CYLINDER*

O. Yu. Tsvelodub

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We consider the flow of a film of viscous fluid on the outside surface of a cylinder of radius R. The Navier–Stokes solutions with film thickness h_0 = const are solved for any fluid flow rate. Such a flow is known to be unstable against infinitesimal perturbations, even for very small Reynolds numbers. Perturbations on the surface of a cylindrical film have been studied in a number of papers (axisymmetric perturbations, as a rule).

Nonlinear interaction between waves of different modes can result in the formation of steady-state traveling-wave modes. Here we consider such nonlinear space waves. A particular class of space-wave modes (helical waves) was studied in [1], where we derived the corresponding model equation. That equation will be described briefly here.

Let us consider the case of large cylinders, for which $\delta = H_0/R \ll 1$. For low flow rates the solution can be sought in the form of a series in the small parameter $h_0/L \ll 1$ (L is the characteristic perturbation length). Then all of the quantities can be represented by polynomials in the transverse coordinate with coefficients that depend only on the film thickness h and its derivatives. Then, using the kinematic condition for a free surface, we can obtain one equation for the film thickness h. If only terms of order up to and including ε^2 are considered, that equation is

$$\frac{\partial h}{\partial t}+\varepsilon\frac{\mathrm{Re}}{\mathrm{Fr}}h^2h_x+\varepsilon^2\left\{S\frac{\mathrm{Re}}{\mathrm{Fr}}\frac{h^3}{3}h_x+\frac{\mathrm{Re}^2}{\mathrm{Fr}}\left[\frac{5}{6}h_th_xh^3-\frac{5}{2}h_{xt}h^4\right]-\right.$$
$$-\frac{\mathrm{Re}^3}{\mathrm{Fr}^2}\left[\frac{9}{20}h_x^2h^5-\frac{3}{40}h^6h_{xx}\right]+\frac{\mathrm{Re}}{\mathrm{Fr}}\mathrm{We}\,\varepsilon^2[(S^2h_x^2+h_x\Delta h_x)h^2+ \tag{1}$$
$$\left.+\frac{1}{3}(S^2h^3\Delta h+h^3\Delta^2h)+S^2h^2h_\varphi\Delta h_\varphi+S^4h_\varphi^2h^2]\right\}=0,$$

where the subscript of h denotes differentiation with respect to the corresponding variable, $\Delta \equiv \partial^2/\partial X^2 + S^2\,\partial^2/\partial\varphi^2$; $\varepsilon = h_0/L$; $S = L/R$; $Re = h_0V_0/\nu$ is the Reynolds number, $Fr = V_0^2/gh_0$ is the Froude number, $\mathrm{We} = \sigma/\rho g h_0^2$ is the Weber number, ν is the viscosity, g is the free fall acceleration, σ is the surface tension, ρ is the density of the fluid, and V_0 is the characteristic velocity when a film of thickness h flows in the waveless mode.

As shown in [1], if the wavelength L of neutral axisymmetric perturbations is taken to be the characteristic longitudinal scale of length and the velocity V_0 on the free surface in the waveless mode of flow is taken to be the characteristic velocity, then for the parameter S we have

$$S\equiv L/R=1/(1+0{,}8\,\mathrm{Re}/\mathrm{We}\,\delta^2)^{1/2}<1.$$

In deriving Eq. (1) from the complete system of Navier–Stokes equations we used the assumptions that

$$h_0/R\ll 1,\qquad \varepsilon=h_0/L\ll 1,$$
$$\mathrm{Re}=V_0h_0/\nu\leqslant 1,\quad \mathrm{We}=\sigma/\rho g h_0^2\gg 1,\quad \mathrm{We}\,\varepsilon^2\sim 1.$$

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Institute of Thermal Physics, Siberian Branch of the Russian Academy of Sciences, 630090 Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 36, No. 6, pp. 77-84, November-December, 1995. Original article submitted December 1, 1994.

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Henceforth the discussion is confined to weakly nonlinear perturbations and the function h is written as

$$h = 1 + \varepsilon h_1.$$

Using the method of different time scales and leaving on the main orders in (1), after some transformations we obtain [1]

$$\frac{\partial H}{\partial \tau} + 4H\frac{\partial H}{\partial X} + \frac{\partial^2 H}{\partial X^2} + S^4\frac{\partial^2 H}{\partial \varphi^2} + \left(\frac{\partial^2}{\partial X^2} + S^2\frac{\partial^2}{\partial \varphi^2}\right)^2 H = 0. \tag{2}$$

Equation (2) has been written in a system that moves with velocity of infinitesimal neutral axisymmetric perturbations. This equation was evidently first obtained in [2]. Other combinations of characteristic quantities were used in the transformations there, it is true, and ultimately a different variable parameter appears in the equation (in second place) instead of S.

Investigation of perturbations in a film flowing down the surface of a vertical cylinder within the limitations used here reduces to analysis of the solutions of Eq. (2).

In the case of axisymmetric solutions ($H = H(\tau, X)$) this equation goes over into a well known equation, often called the Kuramoto–Sivashinskii equation:

$$\frac{\partial H}{\partial \tau} + 4H\frac{\partial H}{\partial X} + \frac{\partial^2 H}{\partial X^2} + \frac{\partial^4 H}{\partial X^4} = 0.$$

If the nonlinear term in (2) is ignored, then it follows from the linearized equation that the trivial solution $H = 0$ is unstable against perturbations of the form

$$\exp(\mathrm{i}\alpha(X - c\tau) + \mathrm{i}n\varphi) \tag{3}$$

with the components of the wave vector (α, n), which satisfy the inequality

$$\alpha^2 + S^4 n^2 - (\alpha^2 + S^2 n^2)^2 > 0, \tag{4}$$

where n and α are natural and real numbers, and $c = c_r + ic_i$ is the complex phase velocity of the perturbations.

The range of the perturbations ($c_i < 0$) is determined by the inequality of the opposite sign ($<$) instead of (4). Accordingly, the wave numbers of neutral perturbations should satisfy

$$\alpha^2 + S^4 n^2 - (\alpha^2 + S^2 n^2)^2 = 0. \tag{5}$$

Solving Eq. (5) for α^2, we obtain

$$\alpha_{1,2}^2 = \{1 - 2S^2 n^2 \pm (1 - 4S^2 n^2 (1 - S^2))^{1/2}\}/2. \tag{6}$$

As is seen from (6), neutral perturbations with a given azimuthal number n ($n \geq 1$) exists for values of S that satisfy

$$S \leqslant S_*(n) = \{[n - (n^2 - 1)^{1/2}]/2n\}^{1/2}. \tag{7}$$

For the values of S and n given in (7) the range of unstable wave numbers α lies inside the interval (α_1, α^2). The general case is when that interval is finite. When the roots become close to each other, which occurs as $S \to S_*(n)$, we have a special case. In those cases the exponential growth of perturbations of the type (3) ceases as a result of nonlinear effects and nonlinear steady-state space-wave modes can arise.

For $S > S_*(n)$ all perturbations of the type (3) are stable. For those values of S and n, therefore, space-wave modes with a finite amplitude cannot arise from the trivial mode $H = 0$ as a result of a nonlinear evolution of infinitesimal perturbations; although it cannot be ruled out that such modes can form when axisymmetric waves of finite amplitude lose stability and then evolve nonlinearly.

We look for a solution for (1) in the form

$$H(\xi, \varphi), \quad \xi = X - c\tau. \tag{8}$$

The following symmetry condition holds for the solutions (8):

$$H(\xi, \varphi) = H(\xi, -\varphi). \tag{9}$$

In accordance with the linear theory of stability, in the plane (α, n) space-wave periodic solutions of the type (9), which have an infinitesimal amplitude, branch off from the trivial solution at neutral points given by Eq. (5). They can be written as

$$H = \Gamma \exp \mathrm{i} \alpha X \{\exp \mathrm{i}\, n\varphi + \exp[-\mathrm{i}\, n\varphi]\} + \text{c.c.}, \tag{10}$$

where c. c. is a complex-conjugate expression.

Solutions for small but finite amplitudes exist in the neighborhood of the neutral points (6). Let us consider weakly nonlinear motions that perturb the free surface only slightly. Our goal is to construct periodic solutions of Eq. (2), which vary slowly with time, as $O(\varepsilon^{-m})$ (the value of m is determined later).

The solutions are written as a series in the small parameter:

$$H = \varepsilon H_0 + \varepsilon^2 H_1 + \varepsilon^3 H_2 + \dots \tag{11}$$

Here, the order of the amplitude of the first harmonic of (10) can serve at the parameter ε.

We introduce a set of fast and slow variables

$$x_n = \varepsilon^n X, \quad t_n = \varepsilon^n \tau, \quad n = 0, 1, 2, \dots$$

Then the differentiation operations in (2) have the form

$$\begin{aligned}
&\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \dots, \\
&\frac{\partial}{\partial X} = \frac{\partial}{\partial x_0} + \varepsilon \frac{\partial}{\partial x_1} + \dots, \\
&\frac{\partial^2}{\partial X^2} = \frac{\partial^2}{\partial x_0^2} + 2\varepsilon \frac{\partial^2}{\partial x_1 \partial x_0} + \varepsilon^2 \Big(\frac{\partial^2}{\partial x_1^2} + 2\frac{\partial^2}{\partial x_2 \partial x_0}\Big) + \dots, \\
&\Big(\frac{\partial^2}{\partial X^2} + S^2 \frac{\partial^2}{\partial \varphi^2}\Big)^2 = \Big(\frac{\partial^2}{\partial x_0^2} + S^2 \frac{\partial^2}{\partial \varphi^2}\Big)^2 + 4\varepsilon \Big(\frac{\partial^4}{\partial x_1 \partial x_0^3} + S^2 \frac{\partial^4}{\partial x_1 \partial x_0 \partial \varphi^2}\Big) + \\
&+ \varepsilon^2 \Big(6\frac{\partial^4}{\partial x_0^2 \partial x_1^2} + 4\frac{\partial^4}{\partial x_2 \partial x_0^3} + 2S^2 \frac{\partial^4}{\partial x_1^2 \partial \varphi^2} + 4S^2 \frac{\partial^4}{\partial x_2 \partial x_0 \partial \varphi^2}\Big) + \dots
\end{aligned} \tag{12}$$

Since Eq. (2) has been written in a reference frame that moves with the velocity of infinitesimal perturbations, we consider solutions for which the functions H_n in (11) depend on x_0 periodically with a spatial period of $2\pi/\alpha$ and do not depend on t_0.

Substituting the series (11) into (2) and collecting terms with the same powers of ε, with allowance for (12), we have an infinite system of linear inhomogeneous equations for different orders of ε.

The equation that corresponds to the first order in that system is

$$\frac{\partial H_0}{\partial t_0} + \frac{\partial^2 H_0}{\partial x_0^2} + S^4 \frac{\partial^2 H_0}{\partial \varphi^2} + \Big(\frac{\partial^2}{\partial x_0^2} + S^2 \frac{\partial^2}{\partial \varphi^2}\Big)^2 H_0 = 0. \tag{13}$$

As already mentioned, in the general case for given components of the wave vector (α, n) its solution is represented by various sums, consisting of terms of the form (3). On substituting the solution of the type (10) that we need, we have

$$\frac{\partial \Gamma}{\partial t_0} - \alpha c_i \Gamma = 0, \tag{14}$$

where $\alpha c_i = \alpha^2 + S^4 n^2 - (\alpha^2 + S^2 n^2)^2$; and c_i is the imaginary part of the complex phase velocity c.

The second term in (14) should be fairly small if Γ is to be assumed to be independent of t_0. We assume that

$$\alpha c_i \Gamma \approx \varepsilon^2. \tag{15}$$

The second term in (14), therefore, should be carried over to a third-order equation. Equation (15) is valid if the wave numbers are close to the neutral values (6).

Corresponding to the second order in ε is the equation

$$\begin{gathered}\frac{\partial H_0}{\partial t_1} + 4H_0\frac{\partial H_0}{\partial x_0} + \frac{\partial^2 H_1}{\partial x_0^2} + S^4\frac{\partial^2 H_1}{\partial \varphi^2} + 2\frac{\partial^2 H_0}{\partial x_0 \partial x_1} + \Big(\frac{\partial^2}{\partial x_0^2} + S^2\frac{\partial^2}{\partial \varphi^2}\Big)^2 H_1 + \\ + 4\Big(\frac{\partial^4}{\partial x_0^3 \partial x_1} + S^2\frac{\partial^4}{\partial x_0 \partial x_1 \partial \varphi^2}\Big) H_0 = 0.\end{gathered} \tag{16}$$

The condition for (16) not to have secular terms is given by

$$\frac{\partial \Gamma}{\partial t_1} + 2\,\mathrm{i}\,\alpha[1 - 2(\alpha^2 + S^2 n^2)]\frac{\partial \Gamma}{\partial x_1} = 0. \tag{17}$$

When the validity of (17) is taken into account, we obtain H_1 from (16),

$$\begin{gathered}H_1 = \Gamma_1 \exp 2\,\mathrm{i}\,\alpha\, x_0 \{\exp 2\,\mathrm{i}\, n\, \varphi + \exp[-2\,\mathrm{i}\, n\, \varphi]\} + \text{c.c.}, \\ \Gamma_1 = -\{\mathrm{i}\,\alpha/[4(\alpha^2 + S^2 n^2)^2 - \alpha^2 - S^4 n^2]\}\Gamma^2.\end{gathered} \tag{18}$$

If periodic solutions of finite amplitude with weak modulation at times of the order of $O(\varepsilon^{-2})$ are to exist we must require that Γ be independent of t_1:

$$\frac{\partial \Gamma}{\partial t_1} = 0. \tag{19}$$

Satisfaction of (19) requires that

$$2\alpha[1 - 2(\alpha^2 + S^2 n^2)]\frac{\partial \Gamma}{\partial x_1} \leqslant O(\varepsilon) \tag{20}$$

for the nontrivial solution Γ.

When the neutral points (n, α_1) and (n, α_2) defined in (6) are separated by a finite interval, the coefficient of the derivative in (20) is finite and the inequality can be satisfied only if Γ does not depend on x_1. In that case, assuming that $\Gamma = \Gamma(t_2, x_2)$, we obtain an equation for the approximation of the third order in ε,

$$\begin{gathered}\underline{\frac{\partial H_0}{\partial t_2}} + \underline{4\Big(H_0\frac{\partial H_1}{\partial x_0} + H_1\frac{\partial H_0}{\partial x_0}\Big)} + \frac{\partial^2 H_2}{\partial x_0^2} + S^4\frac{\partial^2 H_2}{\partial \varphi^2} + \underline{2\frac{\partial^2 H_0}{\partial x_0 \partial x_2}} + \\ + \Big(\frac{\partial^2}{\partial x_0^2} + S^2\frac{\partial^2}{\partial \varphi^2}\Big)^2 H_2 + \underline{4\Big(\frac{\partial^4}{\partial x_0^3 \partial x_2} + S^2\frac{\partial^4}{\partial x_0 \partial x_2 \partial \varphi^2}\Big)H_0} \underline{- \varepsilon^{-2} K H_0} = 0,\end{gathered} \tag{21}$$

where the last term, written in operator form, represents terms carried over from the first approximation. The factor $-\alpha c_i$ appears in from of the amplitude Γ when the operator K acts on the first term in (10). The secular terms are underlined in Eq. (21) (those will be only some terms for the product $H_0 H_1$). The requirement that they be zero for the restricted solution H_2 leads to an equation for the first harmonic of Γ:

$$\frac{\partial \Gamma}{\partial t_2} + \mathrm{i} B \frac{\partial \Gamma}{\partial x_2} - \alpha c_i \varepsilon^{-2} \Gamma + A|\Gamma|^2 \Gamma = 0. \tag{22}$$

Here

$$B = 2\alpha[1 - 2(\alpha^2 + S^2 n^2)]; \quad A = 4\alpha^2/[4(\alpha^2 + S^2 n^2)^2 - (\alpha^2 + S^4 n^2)];$$
$$\alpha c_i = (\alpha^2 + S^4 n^2) - (\alpha^2 + S^2 n^2)^2; \quad t_2 = \varepsilon^2 \tau; \quad x_2 = \varepsilon^2 X.$$

In the special case, when the neutral points (n, α_1) and (n, α_2) specified in (6) are close to each other, the coefficient of the derivative in inequality (20) is small and the inequality can be satisfied even if Γ depends on x_1. Clearly, the second term in (17) is now carried over to the equation of the next order in ε.

Assuming that $\Gamma = \Gamma(t_2, x_1)$, we obtain for the approximation of the third order in ε the equation

$$\begin{aligned} &\underline{\frac{\partial H_0}{\partial t_2}} + 4\Big(H_0 \frac{\partial H_0}{\partial x_1} + \underline{H_0 \frac{\partial H_1}{\partial x_0} + H_1 \frac{\partial H_0}{\partial x_0}}\Big) + \frac{\partial^2 H_2}{\partial x_0^2} + S^4 \frac{\partial^2 H_2}{\partial \varphi^2} + 2\frac{\partial^2 H_1}{\partial x_0 \partial x_1} + \underline{\frac{\partial^2 H_0}{\partial x_1^2}} + \\ &+ \Big(\frac{\partial^2}{\partial x_0^2} + S^2 \frac{\partial^2}{\partial \varphi^2}\Big)^2 H_2 + 4\Big(\frac{\partial^4}{\partial x_0^3 \partial x_1} + S^2 \frac{\partial^4}{\partial x_0 \partial x_1 \partial \varphi^2}\Big) H_1 + \\ &+ \underline{\Big(6\frac{\partial^4}{\partial x_0^2 \partial x_1^2} + 2S^2 \frac{\partial^4}{\partial x_0^2 \partial \varphi^2}\Big)} H_0 - \underline{\varepsilon^{-2} K H_0} + \underline{\varepsilon^{-1} K_1 H_0} = 0, \end{aligned} \tag{23}$$

where the last term represents terms carried over from the second approximation. When the operator K_1 acts on the first term in (10), the amplitude Γ is transformed into the second term in Eq. (17).

$$\frac{\partial \Gamma}{\partial t_2} + \mathrm{i} B \varepsilon^{-1} \frac{\partial \Gamma}{\partial x_1} - \alpha c_i \varepsilon^{-2} \Gamma - P \frac{\partial^2 \Gamma}{\partial x_1^2} + A|\Gamma|^2 \Gamma = 0. \tag{24}$$

Here $P = 6\alpha^2 + 2S^2 n^2 - 1$ and the other coefficients are like those given in (22).

After the transformation

$$\Gamma = \Gamma' \exp(\mathrm{i} \mu x_1), \quad \mu = \varepsilon^{-1} B/2P$$

Eq. (24) becomes (the prime in Γ' is omitted)

$$\frac{\partial \Gamma}{\partial t_2} - \alpha c_i' \varepsilon^{-2} \Gamma - P \frac{\partial^2 \Gamma}{\partial x_1^2} + A|\Gamma|^2 \Gamma = 0, \tag{25}$$

where $\alpha c'_i = \alpha c_i + B^2/4P$. Equation (25) is the familiar Ginzburg–Landau equation.

Equations (22) and (24) have the same spatially homogeneous solutions, which describe the nonlinear stage of the variation of the amplitude for perturbations that increase exponentially in the linear stage:

$$\varepsilon^2 |\Gamma(t_2)|^2 = \frac{\alpha c_i \exp[2\alpha c_i \varepsilon^{-2}(t_2 - t_{20})]}{1 + A \exp[2\alpha c_i \varepsilon^{-2}(t_2 - t_{20})]}. \tag{26}$$

Here t_{20} is a time constant that reflects the arbitrary nature of the assignment of the initial phase of the perturbation.

As $t_2 \to \infty$, we arrive at the steady-state value of the amplitude:

$$\varepsilon |\Gamma_\infty| = (\alpha c_i / A)^{1/2}. \tag{27}$$

Steady-state spatially periodic solutions of finite amplitude, therefore, have thus been obtained for Eqs. (2) in the form of an asymptotic series (11). The first terms of the series are given by Eqs. (10) and (18) while the next term H_2 is easily determined from (22) (or, in the special case, from (24)) but is not given here.

Since the coefficient A for the perturbations under consideration is always greater than zero, clearly waves of finite amplitude are formed from linear perturbations for which $c_i > 0$ as a result of evolution, i.e., weakly nonlinear wave modes are formed in the region of linear instability of the trivial mode. In other words, stability is lost in a soft manner in the given case.

We consider the stability of the solutions (27) to perturbations of "side frequencies" of width $k\varepsilon^j$ ($j = 2$ for Eq. (22), $j = 1$ for Eq. (24)). Here the perturbations are described in the space of ordinary variables x, t. From a comparison of Eqs. (22) and (24) it is clear that in the special case (when α_1 and α_2 are close to each other) the initial periodic solution is modulated more by the short-wave perturbations ($j = 1$) than in the general case ($j = 2$).

When Eq. (22) is valid, we consider the solution ($\delta \ll 1$)

$$\Gamma = \Gamma_\infty + \delta\Gamma_1(t_2)\exp(\mathrm{i}\,kx_2) + \delta\Gamma_2(t_2)\exp(-\mathrm{i}\,kx_2). \tag{28}$$

We substitute (28) into (22), confining the discussion to linear perturbations. Then, equating the coefficients for identical exponents, we obtain

$$\frac{\partial}{\partial t_2}\begin{bmatrix}\Gamma_2\\ \overline{\Gamma_3}\end{bmatrix} + \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\Gamma_2\\ \overline{\Gamma_3}\end{bmatrix} = 0, \tag{29}$$

where

$$A_{11} = \alpha c_i\varepsilon^{-2} - BK, \quad A_{12} = A\Gamma_\infty^2, \quad A_{21} = A\overline{\Gamma}_\infty^2, \quad A_{21} = \alpha c_i\varepsilon^{-2} + BK,$$

and the overbar denotes complex conjugation. By virtue of the linearity of the system (29) its solution has the form

$$\begin{bmatrix}\Gamma_2\\ \overline{\Gamma_3}\end{bmatrix} = \begin{bmatrix}c_2\\ c_3\end{bmatrix}\exp(\gamma t_2) \tag{30}$$

(c_2 and c_3 are constants that are not determined within the framework of linear analysis). Substituting (30) into (29), we arrive at the problem for the characteristic value γ. From the requirement that a nontrivial solution (30) exist we have

$$\gamma_{1,2} = -\alpha c_i\varepsilon^{-2} \pm (\alpha^2 c_i^2\varepsilon^{-4} + B^2k^2)^{1/2}. \tag{31}$$

It is clear from (30) that the condition that the solution (27) be stable against the perturbations (28) requires that $\gamma < 0$. As follows from (31), this condition is not satisfied for every value of k. The solutions obtained thus are unstable against to "side frequency" perturbations with a modulation width of the order $O(\varepsilon^2)$.

In the special case, when Eq. (24) holds, investigation of the stability against perturbations of width $k\varepsilon$ leads to the following relations for γ:

$$\gamma_{1,2} = -Pk^2 - \alpha c_i\varepsilon^{-2} \pm (\alpha^2 c_i^2\varepsilon^{-4} + B^2k^2)^{1/2}.$$

Here, as in (31), the inequality $\gamma_1 < 0$ is always valid, i.e., the perturbation corresponding to that characteristic value is stable. Unlike the case in (31), the second characteristic value γ_1 can now be smaller than zero, i.e., the solution (27) in this case is more stable against perturbations with certain wave numbers. It is easy to understand that "shorter-wave" perturbations, for which

$$k^2 \geqslant k_*^2 = (B^2 - 2P\alpha c_i)\varepsilon^{-2}/P^2. \tag{32}$$

Clearly, Eq. (32) is satisfied if $(B^2 - 2P\alpha c_i) > 0$. Then for perturbations with small k we have $\gamma_2 > 0$.

Analysis shows that in this special case a critical k_* exists for all wave numbers from the interval of instability of the critical solution $\alpha_2 \leq \alpha \leq \alpha_i$. It is a minimum for a solution with a wave number in the middle of the region of linear instability ($\alpha = (\alpha_1 + \alpha_2)/2$) and increases monotonically as α approaches the limits of the region of instability.

In summary, it has been demonstrated that Eq. (2) has spatially periodic steady-state solutions and that in both the general and special cases those solutions are unstable against "side frequency" perturbations with fairly small k. In other words, a periodic wave with wave number α is unstable against perturbations with similar wave numbers $\alpha \pm k\varepsilon^{j}$. In the special case, when the solution (27) becomes stable against perturbations with $|k| \geq |k_*|$, more complex wave modes that can be obtained on the basis of Eq. (24) can be expected to appear. Such more complex wave modes can be expected to appear in the general case as well, but evidently at values of k when Eq. (22) is no longer applicable and a more complex formulation must be considered.

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